

Modified Potra-Pták method to determine the multiple zeros of nonlinear equations

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Abstract

In this paper, we present a third-order iterative method based on Potra-Pták method to compute the approximate multiple roots of nonlinear equations. The method requires two evaluations of the function and one evaluation of its first derivative per iteration and it has the efficiency index equal to $3^{\frac{1}{3}} \approx 1.44225$. We describe the analysis of the proposed methods along with numerical experiments including comparison with existing methods. Moreover, the attraction basins are shown and compared with other existing methods.

Keywords: Multi-point iterative methods, Potra-Pták method, Multiple roots, Basin of attraction.

1 Introduction

Solving nonlinear equations based on iterative methods is a basic and extremely valuable tool in all fields of science as well as economics and engineering. The important aspects related to these methods are order of convergence and number of function evaluations. Therefore, it is favorable to attain the highest possible convergence order with fixed number of function evaluations for each iteration. The aim of the paper is to modify third-order the Potra-Pták method to solve nonlinear equations for multiple zeros with same order of convergence and efficiency index. The efficiency index of an iterative method of order p requiring k function evaluations per iteration is defined by $E(k, p) = \sqrt[p]{p}$, see [10].

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Let α be multi roots of $f(x) = 0$ with multiplicity m i.e., $f^{(i)}(\alpha) = 0$, $i = 0, 1, \dots, m-1$ and $f^{(m)}(\alpha) \neq 0$. If functions $f^{(m-1)}$ and $f^{1/m}$ have only a simple zero at α , any of the iterative methods for a simple zero may be used [2, 17]. The modified Newton method defined in [13] is one of the most well known iterative methods for multiple roots

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)},$$

which converges quadratically.

In the recent years, a wide collection of iterative methods for finding simple roots and multiple roots of nonlinear equations have been presented in some journals, see [8, 11, 15, 16]. In order to improve the convergence of iterative methods for multiple roots, many researchers such as Chun et al. [2], Dong [4], Hansen and Patrick [5], Heydari et al. [7], Osada [9], Victory and Neta [19] proposed various iterative methods when the multiplicity m is known.

This paper is organized as follow: Section 2 is devoted to the construction and convergence analysis of a new method which theoretical proof has been given to reveal the third-order of convergence. In Section 3, different numerical tests confirm the theoretical results and allow us to compare this method with other known methods. Comparisons of attraction basins with other methods are illustrated in this section as well. Finally, a conclusion is provided in Section 4.

2 The method and analysis of convergence

2.1 Extension for multiple roots

In this section, we proceed to develop a new iterative method to find an approximate root for a nonlinear equation $f(x) = 0$ where $f : \mathbb{C} \rightarrow \mathbb{C}$.

Our main aim is to extend Potra-Pták iterative method for multiple roots and build an iterative method using some parameters without any additional evaluations of the function or its derivatives. The Potra-Pták iterative method [12] is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)}, \quad (n = 0, 1, \dots), \end{aligned} \tag{2.1}$$

with the initial approximation of x_0 sufficiently close to x^* . Convergence order of Potra-Pták method for approximating simple zero of nonlinear equations is three, whereas for finding multiple zeros is linear. In order to do so, we add two parameters α and β on the second term of (2.1). So

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{\alpha f(x_n) + f(y_n)}{\beta f'(x_n)}, \end{aligned} \tag{2.2}$$

where α and β are parameters to be chosen for maximal order of convergence.

2.2 Proof of convergence

In what follow, we describe the convergence analysis on iterative method (2.2).

Theorem 1. *Let $x^* \in D$ be a multiple zero of a sufficiently differentiable function $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ for an open interval D with integer multiplicity $m \geq 1$, which includes x_0 as an initial approximation of x^* . Then, method (2.2) has order of three, $\alpha = \frac{(\mu-1)\mu^{m-1}}{m^m}$ and $\beta = \frac{\mu^{m-1}}{m^{m+1}}$ where $\mu = m-1$.*

Proof. Let $e_n := x_n - \alpha$, $e_{n,y} := y_n - \alpha$, $c_i := \frac{m!}{(m+i)!} \frac{f^{(m+i)}(\alpha)}{f^{(m)}(\alpha)}$. Using the fact that $f(\alpha) = 0$, Taylor expansion of f at α yields

$$f(x_n) = e_n^m (c_0 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4), \quad (2.3)$$

and

$$f'(x_n) = e_n^{m-1} (m + (m+1)c_1 e_n + (m+2)c_2 e_n^2 + (m+3)c_3 e_n^3 + O(e_n^4)). \quad (2.4)$$

Hence

$$e_{n,y} = y_n - \alpha = \frac{-1+m}{m} e_n - \frac{c_1}{m^2 c_0} e_n^2 + \frac{-(1+m)c_1^2 + 2mc_0 c_2}{m^3 c_0^2} e_n^3 + O(e_n^4). \quad (2.5)$$

For $f(y_n)$ we have

$$f(y_n) = e_{n,y}^m (c_0 + c_1 e_{n,y} + c_2 e_{n,y}^2 + c_3 e_{n,y}^3) + O(e_{n,y}^4). \quad (2.6)$$

Substituting (2.3)-(2.6) in (2.2), we obtain

$$e_{n+1} = D_1 e_n + D_2 e_n^2 + D_3 e_n^3 + O(e_n^4),$$

where

$$D_1 = 1 - \frac{\alpha + (\mu/m)^3}{\beta m}, \quad D_2 = \frac{(\mu\alpha + (\mu-1)(\mu/m)^m)c_1}{\beta\mu m^2 c_0},$$

and

$$D_3 = \frac{-2c_1^2\alpha(-1+m^2) + 4\alpha\mu c_0 c_2 + c_1^2(\mu/m)^m((6 + (1-2\mu)m) + 2c_0 c_2(1 + m(2\mu-3)))}{2\beta\mu m^3 c_0^2}.$$

Therefor, to provide the order of convergence three, it is necessary to choose $D_i = 0$ ($i = 1, 2$), which gives

$$\alpha = (\mu-1)\frac{\mu^\mu}{m^m}, \quad \text{and} \quad \beta = \frac{\mu^\mu}{m^{m+1}},$$

where $\mu = m-1$ and the error equation becomes

$$e_{n+1} = \frac{(2+m)c_1^2 - 2\mu c_0 c_2}{2m^2 c_0^2} e_n^3 + O(e_n^4).$$

□

Therefore method (2.2) has convergence order three, which proved here.

3 Results and discussions

In this section we check the effectiveness of the modified Potra-Pták method and compare with other existing methods which have the same order of convergence.

3.1 Numerical results

Substituting α and β in the Theorem 1 into (2.2), the modified Potra-Pták is

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n + m \frac{(\mu-1)\mu^\mu f(x_n) - m^\mu f(y_n)}{\mu^\mu f'(x_n)}, \end{aligned} \quad (3.1)$$

where $\mu = m - 1$.

The Osada's method [9], is given by

$$x_{n+1} = x_n - \frac{1}{2}m(m+1) \frac{f(x_n)}{f'(x_n)} + \frac{1}{2}(m-1)^2 \frac{f'(x_n)}{f''(x_n)}. \quad (3.2)$$

The Dong's method [4], is given by

$$\begin{aligned} y_n &= x_n + \sqrt{m} \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - m \left(1 - \frac{1}{\sqrt{m}}\right)^{1-m} \frac{f(y_n)}{f'(x_n)}. \end{aligned} \quad (3.3)$$

The Chun's method [2], is given by

$$\begin{aligned} x_{n+1} &= x_n - \frac{m((2\gamma-1)m+3-2\gamma)}{2} \frac{f(x_n)}{f'(x_n)} \\ &\quad + \frac{\gamma(m-1)^2}{2} \frac{f'(x_n)}{f''(x_n)} - \frac{(1-\gamma)m^2}{2} \frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}. \end{aligned} \quad (3.4)$$

In the numerical experiments of this paper we use $\gamma = -1$.

We test method (2.2) on a number of nonlinear equations. Numerical computations have been carried out using variable precision arithmetic with 100 significant decimal digits in the programming package of Mathematica 8 [6].

$f_1(x) = (\ln(1+x^2) + e^{x^2-3x} \sin x)^6, \quad m=6 \quad x^*=0, \quad x_0=0.3$				
	Method (3.1)	Method (3.2)	Method (3.3)	Method (3.4)
$ x_1 - x^* $	0.656e-1	0.678e-1	0.552e-1	0.736e-1
$ x_2 - x^* $	0.128e-2	0.178e-2	0.511e-3	0.118e-2
$ x_3 - x^* $	0.696e-8	0.247e-7	0.273e-9	0.186e-8
COC	3.0000	3.0000	3.0002	3.0020
ACOC	3.1010	3.0997	3.0914	3.2527
$f_2(x) = (x^3 + \ln(1+x))^7, \quad m=7 \quad x^*=0, \quad x_0=0.2$				
	Method (3.1)	Method (3.2)	Method (3.3)	Method (3.4)
$ x_1 - x^* $	0.880e-2	0.792e-4	0.122e-4	0.987e-4
$ x_2 - x^* $	0.417e-6	0.139e-15	0.742e-19	0.340e-15
$ x_3 - x^* $	0.481e-19	0.918e-39	0.918e-39	0.918e-39
COC	3.0000	3.0000	3.0000	3.0000
ACOC	2.9921	3.0000	3.0000	3.0000
$f_3(x) = (x^6 - 8)^2 \ln(x^6 - 7), \quad m=3 \quad x^*=\sqrt{2}, \quad x_0=1.5$				
	Method (3.1)	Method (3.2)	Method (3.3)	Method (3.4)
$ x_1 - x^* $	0.328e-2	0.414e-2	0.236e-2	0.422e-2
$ x_2 - x^* $	0.914e-6	0.247e-5	0.200e-6	0.591e-5
$ x_3 - x^* $	0.184e-17	0.491e-15	0.117e-16	0.129e-16
COC	3.0000	3.0000	3.0000	3.0000
ACOC	3.0085	3.0067	3.0051	3.0339
$f_4(x) = (\ln(x^2 - x + 1) + 4 \sin(x - 1))^{10}, \quad m=10 \quad x^*=1, \quad x_0=1.2$				
	Method (3.1)	Method (3.2)	Method (3.3)	Method (3.4)
$ x_1 - x^* $	0.102e-2	0.182e-2	0.126e-2	0.102e-2
$ x_2 - x^* $	0.583e-10	0.175e-8	0.393e-9	0.583e-10
$ x_3 - x^* $	0.107e-31	0.155e-26	0.119e-29	0.107e-31
COC	3.0000	3.0000	3.0000	3.0000
ACOC	3.0003	2.9998	2.9999	3.0003
$f_5(x) = \ln^2(x-2)(e^{x-3} - 1) \sin(\pi x/3), \quad m=4 \quad x^*=3, \quad x_0=3.1$				
	Method (3.1)	Method (3.2)	Method (3.3)	Method (3.4)
$ x_1 - x^* $	0.126e-3	0.144e-3	0.890e-4	0.253e-3
$ x_2 - x^* $	0.572e-13	0.106e-12	0.132e-13	0.905e-12
$ x_3 - x^* $	0.531e-41	0.424e-40	0.493e-43	0.414e-37
COC	3.0000	3.0000	3.0000	3.0000
ACOC	3.0003	3.0000	3.0000	3.0000

Table 1: Errors, COC and ACOC for methods (3.1), (3.2), (3.3) and (3.4).

In Table 1, new modified Potra-Pták method (3.1) is compared with the methods (3.2),(3.3) and (3.4) on five nonlinear equations which are illustrated from left to right respectively. In addition

the computational order of convergence is defined by [20]

$$\text{COC} \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|},$$

and the approximated computational order of convergence is defined by [3]

$$\text{ACOC} \approx \frac{\ln |(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}.$$

Table 1 show that new modified Potra-Pták method 3.1 support the established theorem given in the previous section.

3.2 Comparison of attraction basins

In this section, we check the comparison of iterative methods in the complex plane by using basins of attraction. The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points. With attractions basins, the study of dynamic behavior of the rational functions associated to an iterative method gives important information about the convergence and stability of the scheme [8, 15]. Let $G : \mathbb{C} \rightarrow \mathbb{C}$ be a rational mapping on the complex plane. For $z \in \mathbb{C}$, we define its orbit as the set $orb(z) = \{z, G(z), G^2(z), \dots\}$. A point $z_0 \in \mathbb{C}$ is called periodic point with minimal period m if $G^m(z_0) = z_0$, where m is the smallest integer with this property. A periodic point with minimal period 1 is called fixed point. Moreover, a point z_0 is called attracting if $|G'(z_0)| < 1$, repelling if $|G'(z_0)| > 1$, and neutral otherwise. The Julia set of a nonlinear map $G(z)$, denoted by $J(G)$, is the closure of the set of its repelling periodic points. The complement of $J(G)$ is the Fatou set $F(G)$, where the basin of attraction of the different roots lie [1].

We use the basin of attraction for comparing the iteration algorithms. For the dynamical point of view, we take a 256×256 grid of the square $[-3, 3] \times [-3, 3] \in \mathbb{C}$ and assign a color to each point $z_0 \in D$ according to the simple root to which the corresponding orbit of the iterative method starting from z_0 converges, and we mark the point as black if the orbit does not converge to a root, in the sense that after at most 100 iterations it has a distance to any of the roots, which is larger than 10^{-3} . In this way, we distinguish the attraction basins by their color for different methods.

In the following figures, the roots of each function are assigned to different colors. In the basins of attraction, the number of iterations needed to approach the solution is indicated in darker or brighter colors. Black color denotes lack of convergence to any of the roots or convergence to infinity.

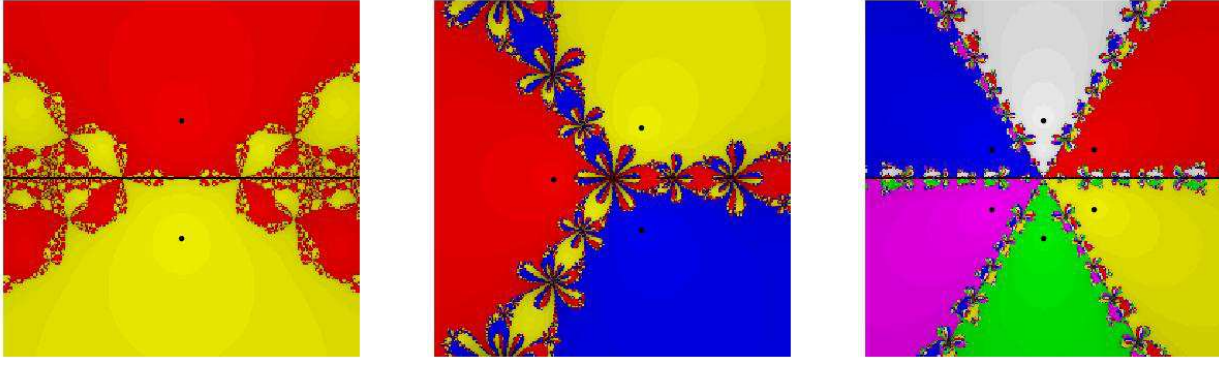


Figure 1: Method (2.1) for test problems $p_1(z) = z + \frac{1}{z}$, $p_2(z) = z^3 + 1$ and $p_3(z) = z^5 + \frac{1}{z}$ respectively, without any multiplicity

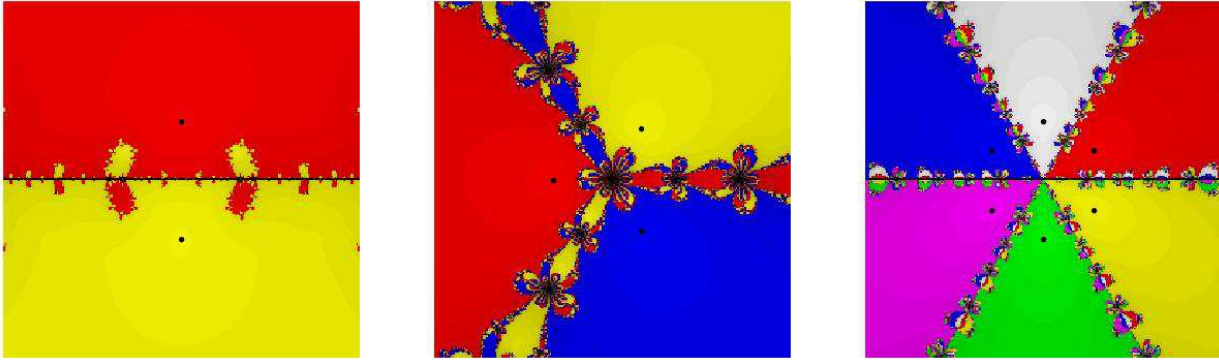


Figure 2: Method (3.1) for test problems $p_1(z) = \left(z + \frac{1}{z}\right)^5$, $p_2(z) = (z^3 + 1)^3$ and $p_3(z) = \left(z + \frac{1}{z}\right)^2$ respectively

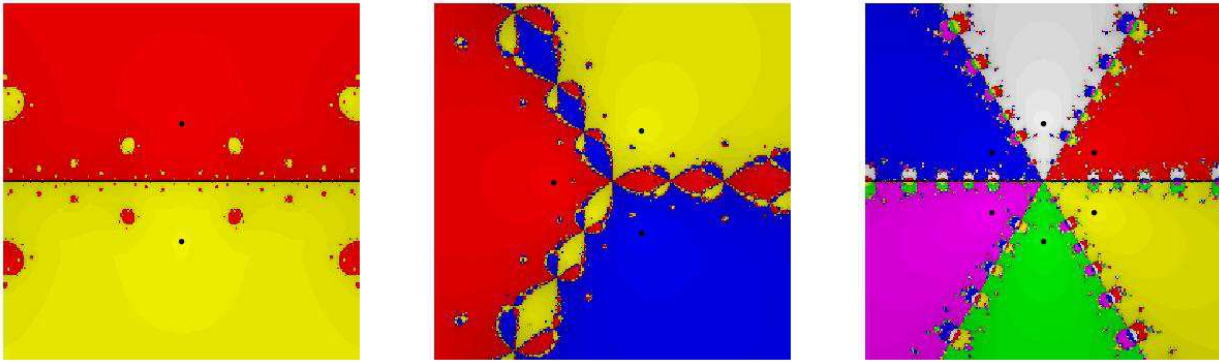


Figure 3: Method (3.2) for test problems $p_1(z) = \left(z + \frac{1}{z}\right)^5$, $p_2(z) = (z^3 + 1)^3$ and $p_3(z) = \left(z + \frac{1}{z}\right)^2$ respectively

In Figures 1-5, basins of attractions of methods (2.1), (3.1), (3.2), (3.3) and (3.4) are illustrated for three test problems $p_1(z)$, $p_2(z)$ and $p_3(z)$ from left to the right respectively. Therefore, the results presented in Table 1 and Figures 1-5 show that our proposed modified Potra-Pták method is competitive to other existing methods.

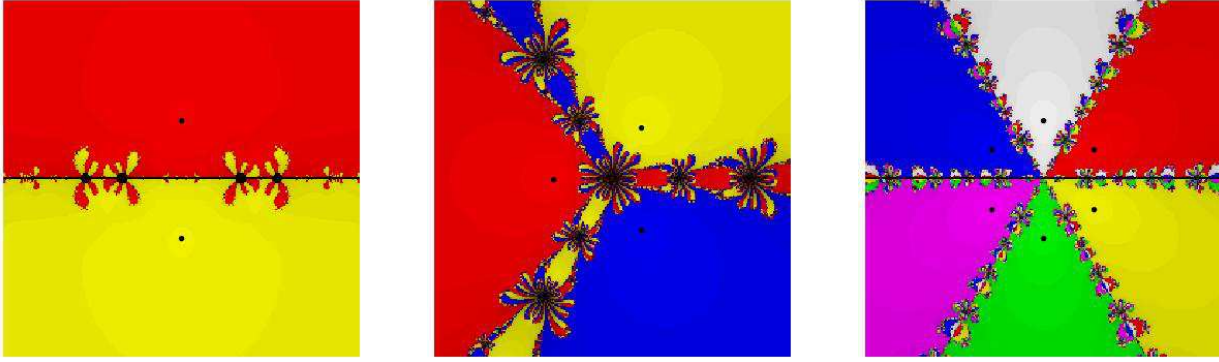


Figure 4: Method (3.3) for test problems $p_1(z) = \left(z + \frac{1}{z}\right)^5$, $p_2(z) = (z^3 + 1)^3$ and $p_3(z) = \left(z + \frac{1}{z}\right)^2$ respectively

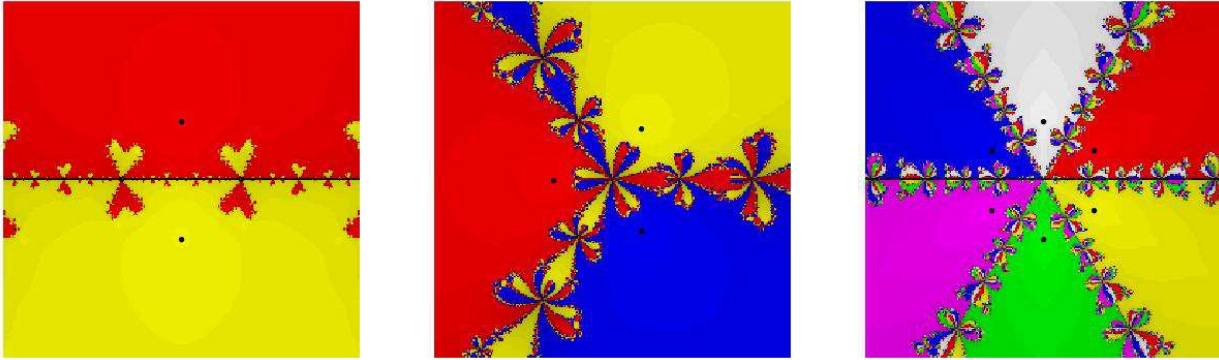


Figure 5: Method (3.4) for test problems $p_1(z) = \left(z + \frac{1}{z}\right)^5$, $p_2(z) = (z^3 + 1)^3$ and $p_3(z) = \left(z + \frac{1}{z}\right)^2$ respectively

4 Conclusion

We have obtained a new method based on Potra-Pták method for approximating multiple roots of non-linear equations with same order of convergence and without any additional evaluations of the function or its derivatives. It has an efficiency index to $3^{\frac{1}{3}} \approx 1.44225$. The theoretical results have been checked with some numerical examples. On the other hand, numerical examples as well as the basin of attraction show that our method works and can compete with other methods in the same class.

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